Chapter 1

Vector and Vector Space

Contents

Vectors can be used by air-traffic controlers when tracking planes, by meteorologists to describe wind conditions, and also it helps to computer programmers to design virtual world. In this chapter, applications of vectors which are commonly used in the study of physics: work, torque and magnetic force will be presented along with the concept of vector and vector space.

1.1 Scaler and Vectors in R^2 and R^3

Definition 1.1.1 A Physical quantities that is described by its magnitude only is called scalar.

Definition 1.1.2 A physical quantities that is described using both magnitude and direction is called vector.

■ **Example 1.1** Temperature, Mass, area, density, volume, etc, are examples of scalars because they are completely described by a number that tells "How Much" like 10^oC and length of 5 m whereas force, displacement, velocity, acceleration, etc are examples of vectors. ■

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Definition 1.1.3 Vectors in R^2 and R^3
A vector in the plane R^2 can be described as $v = (v_1, v_2)$ or $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where $v_1, v_2 \in R$. Similarly, a vector in the space R^3 can be described as a triple of numbers $w = (w_1, w_2, w_3)$ or $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, where $w_1, w_2, w_3 \in R$.

Definition 1.1.4 A number x can be used to represent a point on a line. A pair of numbers or a couple of numbers (x, y) can be used to represent a point in the plane. A triple of numbers (x, y, z) can be used to represent a point in space.

We can say that a single number represents a point in 1-space or on a line, a couple of numbers represents a point in 2-space or on a plane and a triple of numbers represents a point in 3-space or on a space. Although we cannot draw a picture to go further, a quadruple of numbers (x, y, z, w) or (x_1, x_2, x_3, x_4) represent a point in 4-space.

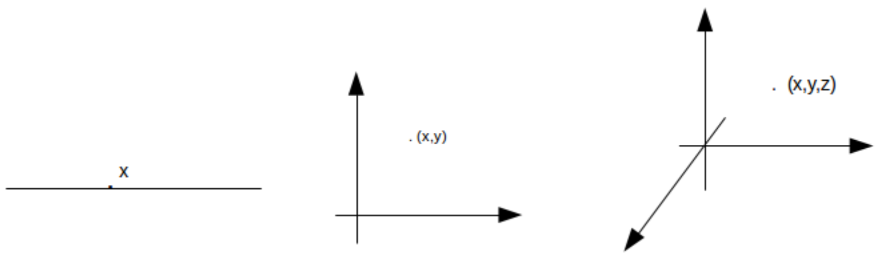

Figure 1.1: Representation of a point on a line, plane and space

Definition 1.1.5 Vectors in n–space,
Every pair of distinct points P and Q in R^n determines a directed line segment with initial point at P and terminal point at Q. We call such a directed line segment a vector and denote it by $\overrightarrow{PQ}$. The length of the line segment is the magnitude of the vector. Although $\overrightarrow{PP}$ has zero length, and strictly speaking, no direction, it is convenient to view it as a vector. It is called a zero or a null vector. It is often denoted by $\overrightarrow{0}$.

Definition 1.1.6 A position vector is a vector whose initial point is at the origin otherwise it is a located vector.

Definition 1.1.7 Two non–zero vectors v and w of the same dimension are said to be parallel if they are scaler multiples of one another. In other words, the two vectors v and w are said to be parallel, denoted by $v//w$ if there is a scaler k such that $v = kw$ and if $k > 0$, then they have the same direction and if $k < 0$, then they are in the opposite direction.

The vector $\overrightarrow{0}$ is parallel to every vector v in the same dimension, since it can be expressed as the scaler multiple $0 = 0v$. Although, zero vectors has no natural direction, it can be assigned any direction that is convenient for the problem at hand.

■ **Example 1.2** Consider $P_1 = (3,7), P_2 = (5,1), Q_1 = (-4,2)$ and $Q_2 = (-16,-14)$ are points on a plane. Then $\overrightarrow{P_1Q_1} = Q_1 - P_1 = (-7,-5)$ and $\overrightarrow{P_2Q_2} = Q_2 - P_2 = (-21,-15) = 3(-7,-5)$. Therefore, $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$ are parallel and have the same direction, since $3 > 0$. ■

■ **Example 1.3** Vector $v = (1,2,3)$ and $w = (-2,-4,-6)$ are parallel vectors because of $w = -2v$, but have opposite direction. ■

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Definition 1.1.8 Two vectors v and w will be considered to be equal(or equivalence), $v = w$, if they have the same magnitude and direction even though they may be located in different position. That is, if $v = (v_1, v_2)$ and $w = (w_1, w_2)$ in R^2, $v = w$ if and only if $v_1 = w_1$ and $v_2 = w_2$.

Figure 1.2: Equal Vectors

The definition of equality of two vectors does not require that the vectors have the same initial and terminal points. Rather it suggests that we can move vectors freely provided we make no change in magnitude and direction.

1.2 Vector Addition and Scaler Multiplication

Definition 1.2.1 If v and w are any two vectors, then the sum $v+w$ is the vector determined as follows; position the vector w so that its initial point coincides with the terminal point of v. The vector $v+w$ is represented by the arrow from the initial point of v to the terminal point of w.

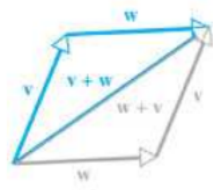

Figure 1.3: The sum of vector v and w

More than two vectors can also be added by joining the terminal point of the first to the initial point of the second and so on, finally the result will be a vector from the initial point of the first to the terminal point of the last vector.

Definition 1.2.2 If v is a non-zero vector and k a non-zero real numbers(scalar), then the product kv is defined to be the vector whose length is $|k|$ times the length of v and whose direction is the same as that of v if $k > 0$ and opposite to that of v if $k < 0$. We define $kv = 0$ if $k = 0$ or $v = 0$.

Note that the vector $(-1)v$ has the same length as v but is oppositely direction. Thus $(-1)v$ is just the negative of v.

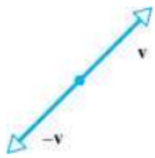

Figure 1.4: -v is in the opposite direction of v

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Vectors in Coordinate System

Let v be any vectors in the plane, that v has been positioned. So, its initial point is at the origin of a rectangular coordinates system. The coordinates (v_1, v_2) of the terminal point of v are called the components of v and we write $v = (v_1, v_2)$. An order pair consists of two terms the abscissa(horizontal, usually x) and the ordinate(vertical, usually y) which define the location of a point in two-dimensional rectangular space.
The operation of vector addition in terms of components for $v = (v_1, v_2)$ and $w = (w_1, w_2)$, then

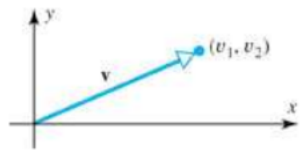

Figure 1.5: The location of a point in two dimensional rectangular space

$$v + w = (v_1 + w_1, v_2 + w_2)$$

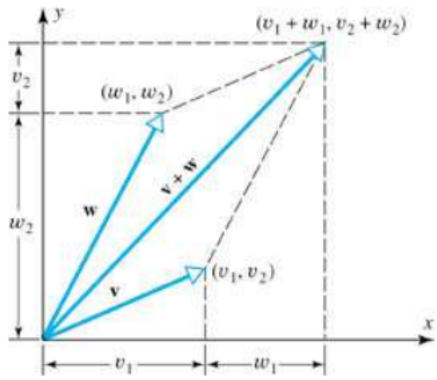

Figure 1.6: The sum of vector v and w component wise

Definition 1.2.3 If $v = (v_1, v_2)$ and k is any number or scaler. Then kv is a vector and defined as $kv = (kv_1, kv_2)$.

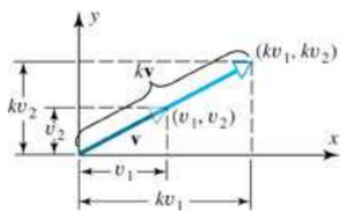

Figure 1.7: Scalar multiple vector

■ **Example 1.4** If $u = (4, 3, 2)$ and $\alpha = 2$, then $\alpha u = 2(4, 3, 2) = (8, 6, 4)$. ■

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If a vector v in 3-space is positioned. So its initial point is at the origin of rectangular coordinate system, the coordinates of the terminal point are called the components of v, and we write $v = (v_1, v_2, v_3)$.

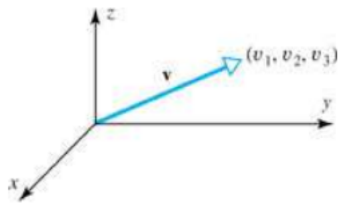

Figure 1.8: Position Vector

Definition 1.2.4 If v and w are any two vectors, then the difference of w from v is defined by

$$v - w = v + (-w)$$

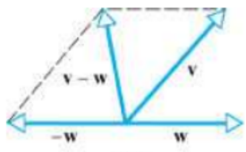

Figure 1.9: The difference of vector v and w

■ **Example 1.5** Consider $v = (1,2,4)$ and $w = (3,1,2)$. Find $v+w$, $2v$ and $v-2w$.
Solution: From the definition of vector addition and scalar multiplication

$$\begin{aligned} v+w &= (1,2,4)+(3,1,2) = (4,3,6) \\ 2v &= 2(1,2,4) = (2,4,8) \\ v-2w &= (1,2,4)-2(3,1,2) = (-5,0,0) \end{aligned}$$

■

Properties of Vector addition and Scaler Multiplication

Let u, v and w be vectors in R^2 and α and β are scalers. Then

1. $v + w \in R^2(R^3)$
2. $v + w = w + v$
3. $u + \overrightarrow{0} = \overrightarrow{0} + u = u$, where $\overrightarrow{0} = (0,0) \in R^2$
4. There exist $w \in R^2$, such that $u + w = \overrightarrow{0}$ for every $u \in R^2$
5. $(u+v)+w = u+(v+w)$
6. $\alpha(\beta u) = (\alpha\beta)u$
7. $(\alpha+\beta)u = \alpha u + \alpha u$
8. $1 \cdot u = u$

The properties described above also hold true for every vectors in R^3, where $\overrightarrow{0} = (0,0,0) \in R^3$ and generally is also true in R^n, where $\overrightarrow{0} = (0,0,0,\cdots,0) \in R^n$.

1.3 Norm of vector and Scalar Product, Orthogonal Projection, and Direction Cosines

1.3.1 Norm of a Vector

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Definition 1.3.1 Let $v = (v_1, v_2)$ be a vector in R^2. Then the norm or magnitude of v, denoted by $\|v\|$ is defined by

$$\|v\| = \sqrt{v_1^2 + v_2^2}$$

Similarly, for a vector $w = (w_1, w_2, w_3)$ be a vector in R^3. Then the magnitude of w, denoted by $\|w\|$ is defined by

$$\|w\| = \sqrt{w_1^2 + w_2^2 + w_3^2}$$

■ **Example 1.6** Find the norm of a vector $u = (2,3,5,4)$.
Solution: From the definition of norm

$$\begin{aligned} \|u\| &= \sqrt{2^2+3^2+5^2+4^2} \\ &= \sqrt{4+9+25+16} \\ &= \sqrt{54} \end{aligned}$$

■

■ **Example 1.7** If $\|v\| = 6$, find x such that $v = (-1, x, 5)$.
Solution: From the definition of norm

$$\begin{aligned} \|v\| &= \sqrt{(-1)^2+x^2+5^2} \\ 6 &= \sqrt{(-1)^2+x^2+5^2} \\ 36 &= 1+x^2+5^2 \\ x^2 &= 10 \\ x &= \pm\sqrt{10} \end{aligned}$$

■

Remark 1.3.1
- $\|v\| \neq 0$ if $v \neq 0$.
- $\|v\| = \|-v\|$

Theorem 1.3.2 If $k \in R$, then $\|kcv\| = |k|\|v\|$.
Proof: Suppose that $v \in R^n$, then

$$\begin{aligned} \|v\| &= \sqrt{(kv_1)^2+(kv_2)^2+\cdots+(kv_n)^2} \\ &= \sqrt{k^2(v_1^2+v_2^2+\cdots+v_n^2)} \\ &= \sqrt{k^2}\sqrt{v_1^2+v_2^2+\cdots+v_n^2} \\ &= |k|\|v\| \end{aligned}$$

■ **Example 1.8** Let $v = (1,3,5)$. Then find the norm or magnitude of the vector $-3v$.
Solution: From the definition and properties of the norm

$$\begin{aligned} \|-3v\| &= |-3|\|v\| \\ &= 3\sqrt{1^2+3^2+5^2} \\ &= 3\sqrt{1+9+25} \\ &= 3\sqrt{35} \end{aligned}$$

■

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Definition 1.3.2 A vector u satisfying $\|u\| = 1$ is called a unit vector.

■ **Example 1.9** The vector $(0,1), (-1,0), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (1,0,0)$ are examples of unit vectors ■

■ **Example 1.10** Find a unit vector in the same direction as $w = (3,-4)$.
Solution: First, note that

$$\|w\| = \|(3,-4)\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

A unit vector in the same direction as w is then $u = \frac{1}{\|w\|} w = \frac{1}{5}(3,-4) = (\frac{3}{5}, \frac{-4}{5})$. ■

■ **Example 1.11** Find a unit vector in the same direction as $(1,-2,3)$ and write $(1,-2,3)$ as the product of its magnitude and a unit vector.
Solution: First, we find the magnitude of the vector

$$\|(1,-2,3)\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$$

The unit vector having the same direction as $(1,-2,3)$ is given by

$$\frac{1}{\sqrt{14}}(1,-2,3) = (\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$$

Furthermore, $(1,-2,3) = \sqrt{14}(\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$. ■

Remark 1.3.3
- All unit vectors in R^2 are of the form $(\cos\theta, \sin\theta)$, where $\theta \in [0, 2\pi]$.
- For any non-zero vector w, the unit vector u corresponding to w in the direction of w can be obtained as $u = \frac{w}{\|w\|}$.
- For two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ on the plane R^3, we calculate the distance $d(P_1, P_2)$ between the two points as

$$d(P_1, P_2) = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

where $\overrightarrow{P_1P_2}$ is the vector with initial point P_1 and terminal point P_2; that is,

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

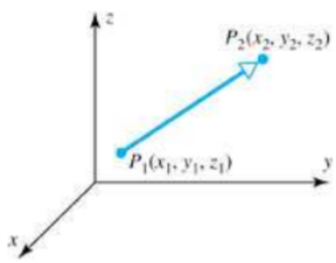

Figure 1.10: Vector

1.3.2 Scalar Product

Definition 1.3.3 Suppose v and w be two vectors in R^2 or R^3 and $\theta \in [0, \pi]$ represents the angle

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between them. Then scalar product of v and w is the number defined by

$$v\cdot w=\begin{cases}\|v\|\|w\|\cos\theta & \text{if } v\neq 0 \text{ and } w\neq 0\\ 0 & \text{if } v=0 \text{ or } w=0\end{cases}$$

The scalar product of the two vectors is a scaler quantity and its value is maximum when $\theta=0^o$ and minimum if $\theta=180^o$ and the scalar product is also called a dot product or inner product and its value are scalar.
Let $v=(v_1,v_2)$ and $w=(w_1,w_2)$ be two non zero vectors. If θ is the angle between v and w, then the law of cosines yields

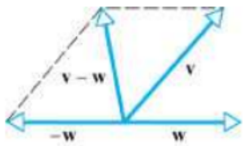

Figure 1.11: The dot product of two vectors

$$\begin{aligned}\|v-w\|^2 &= \|v\|^2+\|w\|^2-2\|v\|\|w\|\cos\theta\\ \Rightarrow 2\|v\|\|w\|\cos\theta &= \|v\|^2+\|w\|^2-\|v-w\|^2\\ &= \|v\|^2+\|w\|^2-\|w-v\|^2\\ \Rightarrow \|v\|\|w\|\cos\theta &= \frac{1}{2}[\|v\|^2+\|w\|^2-\|v-w\|^2]\\ \Rightarrow v\cdot w &= \frac{1}{2}[\|v\|^2+\|w\|^2-\|v-w\|^2]\\ &= \frac{1}{2}[v_1^2+v_2^2+w_1^2+w_2^2+2v_1w_1+2v_2w_2-v_1^2\\ &\quad -v_2^2-w_1^2-w_2^2]\\ &= \frac{1}{2}[2v_1w_1+2v_2w_2]\\ &= v_1w_1+v_2w_2\end{aligned}$$

Therefore, the dot product of the two vectors $v=(v_1,v_2)$ and $w=(w_1,w_2)$ are vectors in R^3, the dot product cab be

$$v\cdot w=v_1w_1+v_2w_2$$

Similarly, if $v=(v_1,v_2,v_3)$ and $w=(w_1,w_2,w_3)$ are non zero vectors in R^3, the dot product can be

$$v\cdot w=v_1w_1+v_2w_2+v_3w_3$$

Properties of Scaler Product

If u,v and w are vectors in the same dimension and $\alpha\in R$, then

1. $u\cdot u=\|u\|^2$
2. $v\cdot w=w\cdot v$
3. $u\cdot(v+w)=u\cdot v+u\cdot w$
4. $0\cdot u=0$
5. $(\alpha v)\cdot w=\alpha(v\cdot w)=w\cdot(\alpha v)$
6. $u\cdot u\geq 0$ and $u\cdot u=0$ if and only if $u=0$

■ **Example 1.12** If $v=(1,-2,3)$ and $w=(0,1,-5)$, then find $v\cdot v$, $v\cdot w$ and $(v+w)\cdot v$.
Solution: From the definition of dot product and properties of dot product

$$\begin{aligned}v\cdot v &= (1,-2,3)\cdot(1,-2,3)=14\\ v\cdot w &= (1,-2,3)\cdot(0,1,-5)=-17\\ (v+w)\cdot v &= ((1,-2,3)+(0,1,-5))\cdot(0,1,-5)\\ &= 9\end{aligned}$$

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■

1.3.3 Angle between two vectors

If θ is the angle between two vectors v and w, then the angle between the two vectors can be obtained by

$$\cos\theta = \frac{v\cdot w}{||v||||w||} \Rightarrow \theta = \cos^{-1}(\frac{v\cdot w}{||v||||w||})$$

where $\theta \in [0,\pi]$

■ **Example 1.13** Find the angle between the vectors $v=(2,0,-2)$ and $w=(2,2,0)$.
Solution: Let θ be the angle between the two vectors, then

$$\cos\theta = \frac{v\cdot w}{||v||||w||} = \frac{4}{\sqrt{8}\cdot\sqrt{8}}$$

From this, $\theta = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$. ■

Definition 1.3.4 Two non-zero vectors v and w are said to be orthogonal(perpendicular) if and only if $v\cdot w = 0$; that is, $\theta = \frac{\pi}{2}$.

■ **Example 1.14** Find the value(s) of x such that the vectors $v=(1,4,3)$ and $w=(x,-1,2)$ are orthogonal.
Solution: From the definition of orthogonality

$$\begin{aligned} v\cdot w &= 0 \\ \Rightarrow &\quad (1,4,3)\cdot(x,-1,2)=0 \\ \Rightarrow &\quad x-4+6=0 \\ \Rightarrow &\quad x=-2 \end{aligned}$$

■

Remark 1.3.4 If v is orthogonal to w, then it is also orthogonal to any scaler multiple of w.

Definition 1.3.5 If P and Q are points in 2 or 3 space, the distance between P and Q, by using dot product, denoted by $||P-Q||$ is given by

$$||P-Q|| = \sqrt{(P-Q)\cdot(P-Q)}$$

Theorem 1.3.5 Given two vectors v and w in space, $||v+w|| = ||v-w||$ if and only if v and w are orthogonal vectors.
Proof:($\Rightarrow$:) Given $||v+w|| = ||v-w||$, we want to show v and w are orthogonal

$$\begin{aligned} ||v+w||^2 &= (v+w)\cdot(v+w) \\ &= ||v||^2+2vw+||w||^2 \\ ||v-w||^2 &= (v-w)\cdot(v-w) \\ &= ||v||^2-2vw+||w||^2 \end{aligned}$$

By hypothesis $||v+w|| = ||v-w||$ implies that $||v||^2+2vw+||w||^2 = ||v||^2-2vw+||w||^2$. From this, $4v\cdot w=0$. Therefore, v and w are orthogonal.
Proof:(:$\Leftarrow$) Given v and w are orthogonal, we want to show $||v+w|| = ||v-w||$

$$\begin{aligned} ||v+w||^2 &= ||v||^2+2vw+||w||^2 \\ ||v-w||^2 &= ||v||^2-2vw+||w||^2 \end{aligned}$$

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since v and w are orthogonal, $v \cdot w = 0$. Therefore, $\|v+w\| = \|v-w\|$.

Theorem 1.3.6 Pythagoras Theorem
If v and w are orthogonal vectors, then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$.
Proof: $\|v+w\|^2$ can be rewritten as

$$\begin{aligned}
\|v+w\|^2 &= (v+w)\cdot(v+w) \\
&= \|v\|^2 + 2v\cdot w + \|w\|^2 \\
&= \|v\|^2 + \|w\|^2, \text{ since } v\cdot w = 0
\end{aligned}$$

■ **Example 1.15** Find any unit vectors that are orthogonal to the vector $v = (6,8)$.
Solution: Let $w = (a,b)$ be a unit vector orthogonal to v, then $\|w\| = 1 = \sqrt{a^2+b^2}$ and $v\cdot w = 6a+8b = 0$.
By using simultaneous equation, $w = (\frac{-4}{5}, \frac{3}{5})$ or $w = (\frac{4}{5}, \frac{-3}{5})$. ■

■ **Example 1.16** If the angle between the vector v and w is $\theta = \frac{\pi}{6}$ with each other and $\|v\| = \sqrt{3}$ and $\|w\| = 1$, then calculate the cosine of the angle between the vectors $v+w$ and $v-w$.
Solution: Let $A = v+w$ and $B = v-w$. Now we need to find the angle between A and B. If ϕ is the angle between A and B, then

$$\begin{aligned}
\cos\phi &= \frac{A\cdot B}{\|A\|\|B\|} \\
&= \frac{(v+w)\cdot(v-w)}{\|v+w\|\|v-w\|} \\
&= \frac{1}{\sqrt{7}} \\
\Rightarrow \phi &= \cos^{-1}(\frac{1}{\sqrt{7}})
\end{aligned}$$

Since

$$\begin{aligned}
\|v+w\|^2 &= \|v\|^2 + 2v\cdot w + \|w\|^2 \\
&= 3 + 2(\sqrt{3})(1)\cos\frac{\pi}{6} + 1 \\
&= 7 \\
\|v+w\| &= \sqrt{7} \\
\|v-w\| &= 1 \\
(v+w)\cdot(v-w) &= \|v\|^2 - \|w\|^2 = 1
\end{aligned}$$

■

■ **Example 1.17** Let v and w be a pair orthogonal vectors such that $\|v\| = t$ and $\|w\| = r$. Find the angle between the vector $p = \frac{tw+rv}{t+r}$ and the vector v.
Solution: Let θ is the angle between the vector p and v. Then

$$\begin{aligned}
\cos\theta &= \frac{p\cdot v}{\|v\|\|p\|} \\
&= \frac{(tw+rv)\cdot v}{(t+r)\|v\|\|p\|} \\
&= \frac{tv\cdot w + r\|v\|^2}{(t+r)t\|p\|} \\
&= \frac{rt}{(t+r)\|p\|} \text{ since } v\cdot w = 0 \text{ and } \|v\| = t
\end{aligned}$$

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But

$$\begin{aligned}
\|p\|^2 &= \frac{\|tw+rv\|^2}{(t+r)^2} \\
&= \frac{t^2\|w\|^2+2tr(v\cdot w)+r^2\|v\|^2}{(t+r)^2} \\
&= \frac{2(tr)^2}{(t+r)^2} \text{ since } v\cdot w = 0 \\
\|p\| &= \frac{\sqrt{2}rt}{t+r}
\end{aligned}$$

Thus, $\cos\theta = \frac{rt}{(t+r)}\frac{(t+r)}{\sqrt{2}rt} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$; that is, $\theta = \cos^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$. ■

1.3.4 Orthogonal Projection

Definition 1.3.6 Suppose S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the magnitude of the vector with representation $\overrightarrow{PS}$ is called the component of w along v and is denoted by $\text{Comp}_v w$; that is,

$$\begin{aligned}
\text{Comp}_v w &= \|w\|\cos\theta \\
&\quad \text{where } \theta \text{ is the angle } v \text{ and } w \\
&= \|w\|\frac{v\cdot w}{\|v\|\|w\|} \text{ since } \cos\theta = \frac{v\cdot w}{\|v\|\|w\|} \\
&= \frac{v\cdot w}{\|v\|}
\end{aligned}$$

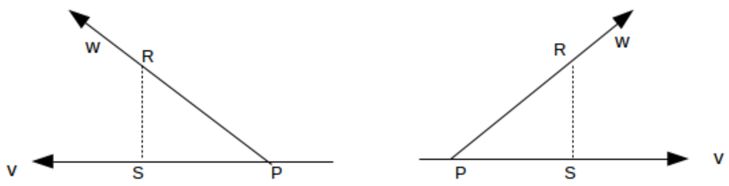

Figure 1.12: Component of w along v

Definition 1.3.7 The projection of w on to v is defined to be the vector w in the direction of vector v, which is denoted by $\text{Proj}_v w$; that is,

$$\begin{aligned}
\text{Proj}_v w &= (\frac{v\cdot w}{\|v\|})\frac{v}{\|v\|} \\
&= \frac{v\cdot w}{\|v\|^2}v
\end{aligned}$$

$$\text{Proj}_v w = (\frac{v\cdot w}{\|v\|})\frac{v}{\|v\|} = \frac{v\cdot w}{\|v\|^2}v$$

■ **Example 1.18** Find the component of v along w and the projection of v on to w, where $v = (1,2)$ and $w = (3,4)$.

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Solution: Since

$$\begin{aligned} \text{Comp}_w v &= \frac{v \cdot w}{\|w\|} \\ &= \frac{1 \times 3 + 2 \times 4}{5} \\ &= \frac{11}{5} \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Proj}_w v &= \left(\frac{v \cdot w}{\|w\|}\right)\frac{w}{\|w\|} \\ &= \frac{11}{5} \cdot \left(\frac{3}{5}, \frac{4}{5}\right) \\ &= \left(\frac{33}{25}, \frac{44}{25}\right) \end{aligned}$$

■

Remark 1.3.7 $\text{Comp}_v w \neq \text{Comp}_w v$ and $\text{Proj}_v w \neq \text{Proj}_w v$

Theorem 1.3.8 Let u be a non-zero vector, then for any other vector w

$$v = w - \frac{w \cdot u}{\|u\|^2} \cdot u$$

is orthogonal to u.
Proof:

$$\begin{aligned} v \cdot u &= \left(w - \frac{w \cdot u}{\|u\|^2} \cdot u\right) \cdot u \\ &= w \cdot u - \frac{w \cdot u}{\|u\|^2}\|u\|^2 \\ &= w \cdot u - w \cdot u \\ &= 0 \end{aligned}$$

■ **Example 1.19** Find an orthogonal vector to $u = (0,2,0,2,1)$.
Solution: Let $w = (0,-1,0,-1,0)$, then

$$\begin{aligned} v &= w - \frac{w \cdot u}{\|u\|^2} \cdot u \\ &= \frac{1}{9}(0,-1,0,-1,0) \end{aligned}$$

is orthogonal(perpendicular) to u. ■

Theorem 1.3.9 Cauchy-Schwarz inequality
For any two vectors v and w

$$v \cdot w \leq \|v\|\|w\|$$

Equality holds if and only if either v is a scalar multiple of w or one of v or w is 0.
Proof: Let p is the end point of $\text{proj}_w v$; that is, $p = \text{proj}_w v$ and let d is the distance from the terminal point of v to the terminal point of the vector $\text{proj}_w v$ from the figure below, $d = \|v - \frac{v \cdot w}{\|w\|^2} w\|$.

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So, from the above assumption, the square of the distance from the line to the origin to be

$$\begin{aligned} \|v - \frac{v\cdot w}{\|w\|^2}w\|^2 &= v\cdot w - 2\frac{(v\cdot w)^2}{w\cdot w} + \frac{(v\cdot w)^2}{w\cdot w} \\ &= \|v\|^2 - \frac{(v\cdot w)^2}{\|w\|^2} \\ &= \frac{1}{\|w\|^2}(\|w\|^2\|w\|^2 - (v\cdot w)^2) \end{aligned}$$

Since the number is a square, it can not be negative. Hence

$$\begin{aligned} (v\cdot w)^2 &\leq \|v\|^2\|w\|^2 \\ \Rightarrow v\cdot w &\leq \|v\|\|w\| \end{aligned}$$

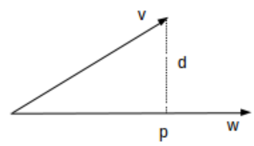

Figure 1.13: The distance from a point on vector v to a point p on vector w

Theorem 1.3.10 Triangular Inequality
For vectors v and w in space

$$\|v+w\| \leq \|v\| + \|w\|$$

Proof: $\|v+w\|^2 = (v+w)\cdot(v+w) = \|v\|^2 + 2v\cdot w + \|w\|^2$
By the cauchy-schwarz inequality, we have

$$\begin{aligned} \|v+w\|^2 &= \|v\|^2 + 2v\cdot w + \|w\|^2 \\ &\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \\ &\leq (\|v\| + \|w\|)^2 \end{aligned}$$

Hence,

$$\|v+w\| \leq \|v\| + \|w\|$$

1.3.5 Directional angles and cosines

Let $u = u_1 i + u_2 j + u_3 k$ be a vector positioned at the origin in R^3, making an angle of α, β and γ with the positive x, y and $z-axis$ respectively. Then the angles α, β and γ are called the directional angles of u, and the quantities $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called the directional cosines of u, which can be computed as

- $\cos\alpha = \frac{u_1}{\|u\|}, \alpha \in [0, \pi]$
- $\cos\beta = \frac{u_2}{\|u\|}, \beta \in [0, \pi]$
- $\cos\gamma = \frac{u_3}{\|u\|}, \gamma \in [0, \pi]$

Remark 1.3.11 $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

1.4 The Vector product

Given two non zero vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, it is very useful to be able to find a non zero vector u that is perpendicular to both v and w. If $u = (u_1, u_2, u_3)$ is such a vector, then $u\cdot v = 0$ and

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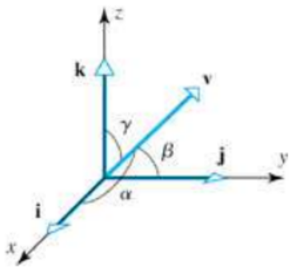

Figure 1.14: Directional angles

$u \cdot w = 0$

$$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 = 0 \quad (1.1)$$

$$u \cdot w = u_1w_1 + u_2w_2 + u_3w_3 = 0 \quad (1.2)$$

To eliminate u_3 multiply (1.1) by w_3 and multiply (1.2) by v_3 and subtract the second equation from the first, then

$$u_1(w_3v_1 - v_3w_1) + u_2(w_3v_2 - v_3w_2) = 0 \quad (1.3)$$

Equation (1.3) has the form $pu_1 + qu_2 = 0$, where the solution for $u_1 = q$ and $u_2 = -p$. So, the solution of equation (1.3) is

$$u_1 = (v_2w_3 - v_3w_2) \text{ and } u_2 = (v_3w_1 - v_1w_3) \quad (1.4)$$

Substituting equation (1.1) and (1.2), we get

$$u_3 = (v_1w_2 - v_2w_1) \quad (1.5)$$

That means a vector perpendicular to both v and w is

$$(u_1, u_2, u_3) = ((v_2w_3 - v_3w_2), (v_3w_1 - v_1w_3), (v_1w_2 - v_2w_1)) \quad (1.6)$$

The resulting vector is called the cross product of v and w denoted by $v \times w$.

Definition 1.4.1 The cross product(or vector product) $v \times w$ of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is defined by

$$v \times w = ((v_2w_3 - v_3w_2), (v_3w_1 - v_1w_3), (v_1w_2 - v_2w_1))$$

Or, $v \times w = n\|v\|\|w\| \sin\theta$, where n is the unit vector in the direction of $v \times w$ and $\theta \in [0, \pi]$ is the angle between v and w.

The cross product of two vectors is a vector.

■ **Example 1.20** Compute $(1,2,3) \times (4,5,6)$.
Solution: From the definition of vector product, we have

$$\begin{aligned}(1,2,3) \times (4,5,6) &= ((2 \times 6 - 3 \times 5), (3 \times 4 - 1 \times 6), (1 \times 5 - 2 \times 4)) \\ &= (-3, 6, -3)\end{aligned}$$

■

Standard Unit Vectors

Consider the vectors

$$i = (1,0,0), j = (0,1,0), k = (0,0,1)$$

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These vectors each have length 1 and lie along the coordinate axis. They are called the standard unit vectors in R^3. Every vector $v = (v_1, v_2, v_3)$ in R^3 is expressed in terms of i, j and k since we can write

$$\begin{aligned} v &= (v_1, v_2, v_3) \\ &= v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1) \\ &= v_1 i + v_2 j + v_3 k \end{aligned}$$

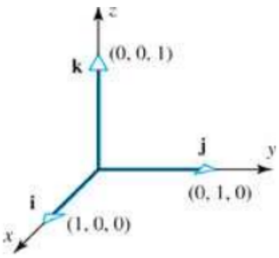

Figure 1.15: Standard unit vectors

■ **Example 1.21** Compute $(2,1,3) \times (1,3,2)$.
Solution: From the definition of vector product, we have

$$\begin{aligned} (2,1,3) \times (1,3,2) &= i(1\times 2 - 3\times 3) - j(2\times 2 - 3\times 1) + k(2\times 3 - 1\times 1) \\ &= -7i - j + 5k \\ &= (-7, -1, 5) \end{aligned}$$

■

Remark 1.4.1 For two non zero vectors v and w,

1. $v \times w$ is a vector which is orthogonal to both v and w
2. $v \times w$ is not defined for $v, w \in R^2$; that is, cross product is defined only in R^3. There is no corresponding operation for vectors in R^2.
3.
   - $i \times j = -(j \times i) = k$ but $i \times i = 0$
   - $j \times k = -(k \times j) = i$ but $j \times j = 0$
   - $k \times i = -(i \times k) = j$ but $k \times k = 0$

   The cross product of two consecutive vectors going clockwise is the next vector around, and the cross product of two consecutive vectors going counterclockwise is the negative of the next vector around.

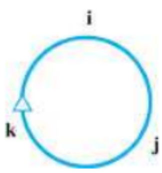

Figure 1.16: The cross product of two consecutive standard unit vectors

Properties of Cross Product

Let u, v and w be vectors in R^3 and α be any scaler. Then

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1. $u \times \overrightarrow{0} = \overrightarrow{0} \times u = \overrightarrow{0}$, where $\overrightarrow{0} = (0,0,0)$
2. $v \times w = -(w \times v)$
3. $u \times (v \times w) \neq (u \times v) \times w$
4. $(\alpha v) \times w = v \times (\alpha w) = \alpha(v \times w)$
5. $u \times (v + w) = (u \times v) + (u \times w)$
6. $v \cdot (v \times w) = w \cdot (v \times w) = 0$
7. If v and w are parallel, then $v \times w = 0$
8. $\|v \times w\| = \|v\|\|w\| \sin\theta, \theta \in [0,\pi]$
9. $\|v \times w\|^2 = \|v\|^2\|w\|^2 - (v \cdot w)^2$
10. $v \times w = n\|v\|\|w\| \sin\theta$, where n is the unit vector in the direction of v and w and $\theta \in [0,\pi]$ is the angle between v and w.

Remark 1.4.2 The angle θ between v and w can be obtained by $\sin\theta = \frac{\|v \times w\|}{\|v\|\|w\|}$ for two non zero vectors v and w.

Definition 1.4.2 Let u, v and w be vectors in R^3. Their scaler triple product is given by

$$u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v)$$

which is a scalar.

Application of Cross Product

1. **Area**
   Let v and w be vectors and consider the parallelogram that the two vectors make. Then the area of the Parallelogram is given by

$$\|v \times w\| = \|v\|\|w\|| \sin\theta|$$

   Since

$$\begin{aligned} \text{Area} &= (\text{Base})(\text{Altitude}) \\ &= \|v\|\|w\| \sin\theta \\ &= \|v \times w\| \end{aligned}$$

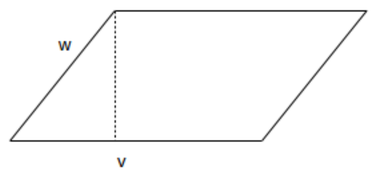

Figure 1.17: Area formed by vector v and w

Remark 1.4.3 The direction of $v \times w$ is a right angle to the parallelogram that follows the right hand rule.

■ **Example 1.22** Find the area of the parallelogram with two adjacent sides formed by the vectors $v = (2,1,3)$ and $w = (4,2,3)$.

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Solution: From the definition of vector product, we have

$$\begin{aligned} v \times w &= (2,1,3)\times(4,2,3) \\ &= i(1\times 3-2\times 3)-j(2\times 3-3\times 4)+k(1\times 2-4\times 1) \\ &= -3i+6j-2k \\ \|v\times w\| &= \sqrt{3^2+6^2+(-2)^2} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

Therefore, the area of the parallelogram is given by 7. ∎

Remark 1.4.4 The area of the triangle formed by v and w as its adjacent sides is given by Area $= \frac{1}{2}\|v\times w\|$

∎ **Example 1.23** Let $u=(-1,\sqrt{3},0)$, $v=(1,-1,1)$ and $w=(0,1,-2)$ be vectors in space. Then find

(a) The unit vector in the opposite direction of u.
(b) The area of the parallelogram formed by u and w.
(c) The angle between $u\times v$ and $u-v$.

Solution:

(a) The unit vector in the opposite direction of u is given by

$$\frac{-u}{\|u\|}=(\frac{1}{2},-\frac{-\sqrt{3}}{2},0)$$

(b) The area of the parallelogram is given by

$$Area=\|u\times w\|=\sqrt{6}$$

(c) Since the $u\times v$ is orthogonal to both u and v, it also orthogonal to $u-v$

∎

∎ **Example 1.24** If the area of an equilateral triangle with adjacent sides v and w is $50\sqrt{3}cm^2$, then find $v\cdot w$.

Solution: Since it is an equilateral triangle, the angle between v and w is $\theta=\frac{\pi}{3}$. The area A of the triangle is

$$\begin{aligned} \text{Area} &= \frac{1}{2}\|v\times w\| \\ &= \frac{1}{2}\|v\|\|w\|\sin\frac{\pi}{3} \\ &= \frac{1}{2}\|v\|\|w\|\frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{4}\|v\|\|w\| \\ \Rightarrow \|v\|\|w\| &= \frac{4}{\sqrt{3}}\text{Area} \\ &= \frac{4}{\sqrt{3}}50\sqrt{3} \\ &= 200 \end{aligned}$$

Therefore,

$$\begin{aligned} v\cdot w &= \|v\|\|w\|\cos\frac{\pi}{3} \\ &= 200(\frac{1}{2})=100 \end{aligned}$$

∎

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■ **Example 1.25** Find the area of the triangle having vertex at $P=(3,-2,-1)$, $Q=(1,3,2)$ and $S=(-2,1,3)$.
Solution: Let $v=\overrightarrow{PQ}=(-2,5,3)$ and $w=\overrightarrow{PS}=(-5,3,4)$, then

$$\begin{aligned}\text{Area} &= \frac{1}{2}\|v\times w\| \\ &= \frac{1}{2}\|(-11,-7,19)\| \\ &= \frac{1}{2}\sqrt{531}\end{aligned}$$

■

2. **Distance**
Let d represent the distance from the point Q to the line through the points P and R. From the elementary trigonometric, we have

$$d=\|\overrightarrow{PQ}\|\sin\theta$$

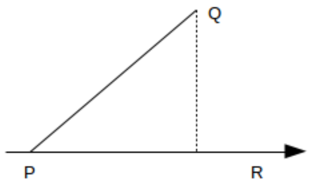

Figure 1.18: The distance from point Q to the vector PR

Where θ is the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Again, we have

$$\begin{aligned}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| &= \|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|\sin\theta \\ &= \|\overrightarrow{PR}\|d\end{aligned}$$

Solving this for d, we get

$$d=\frac{\|\overrightarrow{PQ}\times\overrightarrow{PR}\|}{\|\overrightarrow{PR}\|}$$

■ **Example 1.26** Find the distance from the point $Q=(2,1,3)$ to the line through the point $P=(1,3,2)$ and $R=(1,4,3)$.
Solution: First, we need position vectors corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$; that is, $\overrightarrow{PQ}=(1,-2,1)$ and $\overrightarrow{PR}=(0,1,1)$. So,

$$\begin{aligned}\overrightarrow{PQ}\times\overrightarrow{PR} &= (1,-2,1)\times(0,1,1) \\ &= i(-2\times1-1\times1)-j(1\times1-1\times0)+k(1\times1-(-2)\times0) \\ &= -3i-j+k\end{aligned}$$

We then have

$$\begin{aligned}d &= \frac{\|\overrightarrow{PQ}\times\overrightarrow{PR}\|}{\|\overrightarrow{PR}\|} \\ &= \frac{\sqrt{11}}{\sqrt{2}} \\ &= \sqrt{\frac{11}{2}}\end{aligned}$$

■

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3. **Volume**
For any three non-coplanar vectors u, v and w. Consider the parallelepiped formed using the vectors as three adjacent edges.

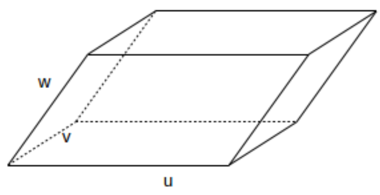

Figure 1.19: Volume formed by vectors u, v and w

The volume of such a solid is given by

$$\text{Volume} = (\text{Area of base})(\text{Altitude})$$

Further, since two adjacent sides of the base are formed by the vector u and v, we know that the area of the base is given by $\|u \times v\|$ and the altitude is given by

$$|\text{Comp}_{u\times v} w| = |\frac{w \cdot (u \times v)}{\|u \times v\|}|$$

The volume of the parallelepiped is then

$$\begin{aligned} \text{Volume} &= \|u \times v\| \cdot \left| \frac{w \cdot (u \times v)}{\|u \times v\|} \right| \\ &= |w \cdot (u \times v)| \end{aligned}$$

Therefore, the volume of the parallelepiped spanned by three vectors u, v and w in R^3 is given by the triple product:

$$\text{Volume} = |u \cdot (v \times w)| = |v \cdot (w \times u)| = |w \cdot (u \times v)|$$

■ **Example 1.27** Find the volume of the parallelepiped with the vectors $u = (1,0,1)$, $v = (2,1,4)$ and $(0,1,1)$ as three of its edges.
Solution: The volume of the parallelepiped V is the absolute value of the triple scalar product of the three vectors u, v and w. Thus,

$$V = |u \cdot (v \times w)| = 1\text{cubic unit}$$

■

■ **Example 1.28** Find the volume of the parallelepiped with three adjacent edges formed by the vectors $u = (1,2,3)$, $v = (4,5,6)$ and $w = (7,8,0)$.
Solution: The volume of the parallelepiped V is the absolute value of the triple scalar product of the three vectors u, v and w. Thus,

$$\begin{aligned} V &= |u \cdot (v \times w)| = 1\text{cubic unit} \\ &= |7(2\times 6 - 5\times 3) - 8(1\times 6 - 4\times 3) + 0(1\times 5 - 2\times 4)| \\ &= |27| \\ &= 27 \end{aligned}$$

Therefore, the volume of the parallelepiped is 27. ■

1.5 Lines and planes in R^3

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Definition 1.5.1 A vector $v = (a,b,c)$ is said to be parallel to a line ℓ if v is parallel to $\overrightarrow{P_0P_1}$ for any two distinct points P_0 and P_1 on l.

Equation of a line

Let $P_0 = (x_0, y_0, z_0)$ be a given point on a line ℓ and $P = (x,y,z)$ be any arbitrary point on l. If $v = (a,b,c)$ is parallel to l, then

1. The parametric equation of a line ℓ is given by
   - $x = x_0 + at$
   - $y = y_0 + bt$
   - $z = z_0 + ct, t \in R$, where t is called the parameter.

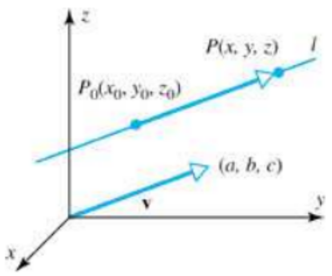

Figure 1.20: The line ℓ parallel to vector v

2. The symmetric equation of a line ℓ is given by

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

   Here, in the equation of a line, a, b or c may be zero but it doesn't mean undefined. For instance $b = 0$, the symmetric equation of a line ℓ is given by

$$\frac{x-x_0}{a} = \frac{z-z_0}{c}, y = y_0$$

   and the line ℓ is parallel to xz–plane.
3. The vector equation of ℓ is written as

$$r - r_0 = tv$$

   where $t \in R$, $r_0 = x_0 i + y_0 j + z_0 k$ and $r = xi + yj + zk$.

■ **Example 1.29** Find the parametric and symmetric equation of the line passing through the point $(2,3,-4)$ and parallel to the vector $(-1,2,5)$.
Solution: Let $P = (x,y,z)$ be any arbitrary point on a line ℓ which contain a point $P_0 = (2,3,-4)$. So $\overrightarrow{P_0P} = tv$ where v is a vector parallel to ℓ and t is some parametric value.

$$\begin{aligned} \overrightarrow{P_0P} &= tv \\ \Rightarrow (x-2, y-2, z+4) &= t(-1,2,5) \\ &= (-t, 2t, 5t) \\ \Rightarrow x-2 &= -t \\ y-2 &= 2t \\ z+4 &= 5t \end{aligned}$$

$$\Rightarrow \quad \left.\begin{array}{r} x = 2-t \\ y = 2+2t \\ z = -4+5t \end{array}\right\} \text{Parametric form of } \ell$$

From parametric form of equation of a line we have symmetric form of the equation of a line. Hence

$$\frac{x-2}{-1} = \frac{y-2}{2} = \frac{z+4}{5}$$

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is symmetric form of the equation of a line. ■

■ **Example 1.30** Find equation of a line through $P_1 = (0,1,2)$ and $P_2 = (-1,1,1)$.
Solution: Let $P = (x,y,z)$ be any arbitrary point on a line ℓ, then the vector $\overrightarrow{P_1P}$ is parallel to the vector $\overrightarrow{P_1P_2} = v$; that is

$$\overrightarrow{P_1P} = tv$$

Since, a point P_1 on the line ℓ and a vector v parallel to the vector formed by two point of the line. Take P_1 and $v = P_2 - P_1$.

$$\begin{aligned} \overrightarrow{P_1P} &= tv \\ \Rightarrow (x-0, y-1, z-2) &= t(-1,0,-1) \\ &= (-t,0,-t) \\ \Rightarrow x &= -t \\ y-1 &= 0 \\ z-2 &= -t \\ \Rightarrow & \left.\begin{array}{r} x=-t \\ y=1 \\ z=2-t \end{array}\right\} \text{Parametric form of } \ell \end{aligned}$$

From the parametric form of the equation of the line by giving distinct values for t we will obtain distinct points on the line and from the parametric form of the equation of the line we will have a symmetric form of the equation of a line

$$\begin{aligned} & x = t(-1), y-1 = t(0), z-2 = t(-1) \\ \Rightarrow \quad & \frac{x}{-1} = \frac{y-1}{0} = \frac{z-2}{-1} = t \\ \Rightarrow \quad & \frac{x}{-1} = \frac{y-1}{0} = \frac{z-2}{-1} \end{aligned}$$

is a symmetric form of the equation of a line. Here in the equation of a line, $\frac{y-1}{0}$ doesn't mean it is undefined but it means the line is parallel to xz plane. ■

■ **Example 1.31** Find an equation of a straight line passing through the points $P_0 = (1,2,-1)$ and $Q_0 = (5,-3,5)$.
Solution: First, we need to find a vector that is parallel to the given line. The obvious choice is

$$v = \overrightarrow{P_0Q_0} = (5-1, -3-2, 4-(-1)) = (4,-5,5)$$

Picking either point will give equation for the line and let $P = (x,y,z)$ be any arbitrary point on ℓ, then $P_0P = tv$; that is,

$$\begin{aligned} \overrightarrow{P_1P} &= tv \\ \Rightarrow (x-1, y-2, z+1) &= t(4,-5,5) \\ &= (4t,-5t,5t) \\ \Rightarrow x-1 &= 4t \\ y-2 &= -5t \\ z+1 &= 5t \\ \Rightarrow & \left.\begin{array}{r} x=1+4t \\ y=2-5t \\ z=-1+5t \end{array}\right\} \text{Parametric form of } \ell \end{aligned}$$

Similarly, the symmetric equation of the line is

$$\frac{x-1}{4} = \frac{y-2}{-5} = \frac{z+1}{5}$$

■

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■ **Example 1.32** Find the equation of a line that contains the point $(1,4,-1)$ and parallel to $v=(-2,3,0)$. Solution: Let $P=(x,y,z)$ be any arbitrary point on a line ℓ which contain a point $P_0=(1,4,-1)$. So, $\overrightarrow{P_0P}=tv$ where v is a vector parallel to ℓ and t is some parametric value.

$$\begin{aligned}\overrightarrow{P_0P} &= tv\\ \Rightarrow (x-1,y-4,z+1) &= t(-2,3,0)\\ &= (-2t,3t,0)\\ \Rightarrow x-1 &= -2t\\ y-4 &= 3t\\ z+1 &= 0t\\ \Rightarrow\quad & \left.\begin{array}{c} x=1-2t\\ y=4+3t\\ z=1\end{array}\right\}\text{ Parametric form of }\ell\end{aligned}$$

From parametric form of equation of a line we have symmetric form of the equation of a line. Hence

$$\frac{x-1}{-2}=\frac{y-4}{3}=\frac{z-1}{0}$$

or

$$\frac{x-1}{-2}=\frac{y-4}{3},z=1$$

is symmetric form of the equation of a line. ■

■ **Example 1.33** Determine if the following two lines are parallel or identical

$$\begin{aligned}\ell_1 :&\quad x=4+t,\ y=3-2t,\ z=2+t \text{ and}\\ \ell_2 :&\quad x=3-2s,\ y=2+4s,\ z=4-2s\end{aligned}$$

Where t and s are some parameter values.
Solution: First look at the direction vectors:

$$v_1=(1,-2,1) \text{ and } v_2=(-2,4,2)$$
$$\Rightarrow v_2=-2v_1$$

Which shows the two lines are parallel. Now, we must determine if they are identical. So, we need to determine if they pass through the same points. So we need to determine if the two sets of parametric equations produce the same points for different values of t and s. Let $t=0$ for line ℓ_1, the point produced is $(4,3,2)$. Set the x from line ℓ_2 equal to the x-coordinate produced by line ℓ_1 and solve for s.

$$4=3-2s\Rightarrow s=-\frac{1}{2}$$

Now, let $s=-\frac{1}{2}$ for line ℓ_2 and the point $(4,0,-1)$ is produced. Since the y and z–coordinates are not equal, the lines are not identical. ■

■ **Example 1.34** Determine if the lines intersect. If so, find the point of intersection and the cosine of the angle of intersection.

$$\begin{aligned}\ell_1 :&\quad x=3+2t,\ y=-2t,\ z=4-t \text{ and}\\ \ell_2 :&\quad x=4-s,\ y=3+5s,\ z=2-s\end{aligned}$$

Where t and s are some parameter values.
Solution: Direction vectors is given by

$$v_1=(2,-2,-1) \text{ and } v_2=(-1,5,-1)$$
$$\Rightarrow v_2\neq kv_1$$

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Therefore those two lines are not parallel. Thus they are either intersect or they are skew lines. Keep in mind that the lines may have a point of intersection or a common point, but not necessarily for the same value of t and s. So, equate each coordinate.

$$\left.\begin{array}{r} x: 3+2t = 4-s \\ y: -2t = 3+5s \\ z: 4-t = 2-s \end{array}\right\} \text{System of 3 equation with 2 unknowns}$$

Solve the first two equation and check with the third equation. So, $t=1$ and $s=-1$. Therefore, ℓ_1 at $t=1$ produces $(5,-2,3)$ and ℓ_2 at$s=-1$ produces the point $(5,-2,3)$. So the lines intersect at this point. To find the angle of intersection of the two lines,

$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\|\|v_2\|}$$

where θ is the angle between v_1 and v_2. The angle θ between two intersecting lines should be less than 90, so we use absolute value in the numerator.

$$\begin{aligned} \cos\theta &= \frac{v_1 \cdot v_2}{\|v_1\|\|v_2\|} \\ &= \frac{|(2,-2,-1)\cdot(-1,5,-1)|}{\|(2,-2,-1)\|\|(-1,5,-1)\|} \\ &= \frac{11}{9\sqrt{3}} \end{aligned}$$

■

Equation of a plane

A plane in space is determine by a point $P_0 = (x_0, y_0, z_0)$ in the plane and a vector $n = (a,b,c)$ is orthogonal to the plane. This orthogonal vector n is called a normal vector.
Suppose that $P = (x,y,z)$ be any arbitrary point in the plane, then

$$\begin{aligned} & n \cdot (\overrightarrow{P_0P}) = 0 \\ & \Rightarrow (a,b,c)\cdot(x-x_0, y-y_0, z-z_0) = 0 \\ & ax+by+cz-(ax_0+by_0+cz_0) = 0 \\ & \Leftrightarrow ax+by+cz+d = 0, \text{ where } d = -(ax_0+by_0+cz_0) \end{aligned}$$

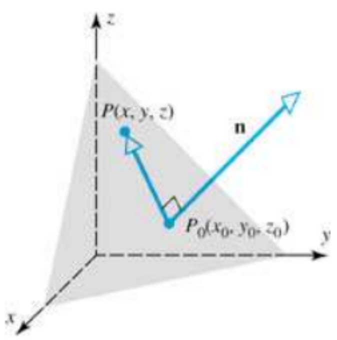

Figure 1.21: The normal vector n to the plane

■ **Example 1.35** Find equation of a plane that contains point $(-2,4,3)$ and is normal to $(3,0,-2)$.
Solution: Let $P = (x,y,z)$ be any arbitrary point in the plane which contain a point $P_0 = (-2,4,3)$ and

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having the normal vector $n = (3,0,-2)$. So, the vector $\overrightarrow{P_0P}$ is orthogonal to n, since n is orthogonal to the plane; that is,

$$n \cdot \overrightarrow{P_0P} = 0$$

So

$$\begin{aligned}
&n \cdot \overrightarrow{P_0P} = 0\\
&(3,0,-2) \cdot (x+2, y-4, z-3) = 0\\
&3(x+2) + 0(y-4) - 2(z-3) = 0\\
&3x + 6 + 0 - 2z + 6 = 0\\
&3x - 2z + 12 = 0
\end{aligned}$$

Therefore, $3x - 2z + 12 = 0$ is the equation of the plane. ■

■ **Example 1.36** Find equation of a plane that contains point $(-1,2,3)$ and is perpendicular to the y-axis. Solution: Let $P = (x,y,z)$ be any arbitrary point in the plane which contain a point $P_0 = (-1,2,3)$ and having the normal vector $n = (0,1,0)$ since y-axis is the normal vector to the plane. So, the vector $\overrightarrow{P_0P}$ is orthogonal to n, since n is orthogonal to the plane; that is,

$$n \cdot \overrightarrow{P_0P} = 0$$

So

$$\begin{aligned}
&n \cdot \overrightarrow{P_0P} = 0\\
&(0,1,0) \cdot (x+1, y-2, z-3) = 0\\
&0(x+2) + 1(y-2) + 0(z-3) = 0\\
&y - 2 = 0\\
&y = 2
\end{aligned}$$

Therefore, $y = 2$ is the equation of the plane which is parallel to xy-plane. ■

■ **Example 1.37** Find the plane containing the three points $P = (1,2,2)$, $Q = (2,-1,4)$ and $R = (3,5,-2)$. Solution: First, we will need to find a vector normal to the plane. Here, the two vectors lying in the plane are $\overrightarrow{PQ} = (1,-3,2)$ and $\overrightarrow{QR} = (1,6,-6)$. Consequently, one vector orthogonal to both of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ is the cross product

$$\overrightarrow{PQ} \times \overrightarrow{QR} = (6,8,9)$$

Since $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are not parallel, $\overrightarrow{PQ} \times \overrightarrow{QR}$ must be orthogonal to the plane containing the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. Now, let $P = (x,y,z)$ be any arbitrary point in the plane which contain a point $P_0 = (1,2,2)$ and having the normal vector $n = (6,8,9)$. So, the vector $\overrightarrow{P_0P}$ is orthogonal to n, since n, the cross product result, is orthogonal to the plane; that is,

$$n \cdot \overrightarrow{P_0P} = 0$$

So

$$\begin{aligned}
&n \cdot \overrightarrow{P_0P} = 0\\
&(6,8,9) \cdot (x-1, y-2, z-2) = 0\\
&6(x-1) + 8(y-2) + 9(z-2) = 0\\
&6x - 6 + 8y - 16 + 9y - 18 = 0\\
&6x + 18y + 9y - 40 = 0\\
&6x + 18y + 9y = 40
\end{aligned}$$

Therefore, $6x + 18y + 9y = 40$ is the equation of the plane. ■

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Definition 1.5.2 Two planes are said to be parallel if their normal vectors are parallel. They are said to be perpendicular if their normal vectors are perpendicular. The angle between two planes is defined to be the angle between their normal.

Example 1.38 Consider the planes

$$\pi_1 : x+2y-3z=2 \text{ and}$$
$$\pi_2 : 15x-9y-z=2$$

Determine the above two planes are parallel or perpendicular.
Solution: Here n_1 for the plane π_1 is $(1,2,-3)$ and n_2 for the plane π_2 is $(15,-9,-1)$. So

$$\begin{aligned} n_1 \cdot n_2 &= (1,2,-3)\cdot(15,-9,-1) \\ &= 15-18+3 \\ &= 0 \end{aligned}$$

This implies that plane π_1 and π_2 are perpendicular. ∎

Example 1.39 Find the cosine of the angle θ between the plane

$$\begin{aligned} \pi_1 : 2x+4y-z &= 0 \text{ and} \\ \pi_2 : x-y+2z &= 2 \end{aligned}$$

Solution: $n_1=(2,4,-1)$ and $n_2=(1,-1,2)$ are the normal vectors of π_1 and π_2, respectively. Let θ be the angle between π_1 and π_2. Then θ is the angle between n_1 and n_2

$$\begin{aligned} \cos\theta &= \frac{n_1\cdot n_2}{\|n_1\|\|n_2\|} \\ &= \frac{2+(-4)+(-2)}{\sqrt{4+16+1}\sqrt{1+1+4}} \\ &= \frac{-4}{\sqrt{126}} \end{aligned}$$

∎

Example 1.40 Find the line of intersection for the planes $\pi_1 : x+3y+4z=0$ and $\pi_2 : x-3y+2z=0$.
Solution: To find the common intersection, solve the equations simultaneously. Multiply the first equation by -1 and add the two to eliminate x; that is,

$$\begin{aligned} -x-3y-4z&=0 \\ x-3y+2z&=0 \\ \Rightarrow -6y-2z=0 \text{ or } y&=-\frac{1}{3}z \end{aligned}$$

Back substitute y into one of the first equations and solve for x.

$$\begin{aligned} x+3(-\frac{1}{3}z)+4z&=0 \\ x-z+4z&=0 \\ x&=-3z \end{aligned}$$

Finally if you late $z=t$, the parametric equations for the lines are

$$x=-3t,\ y=-\frac{1}{3}t,\ z=t$$

where t is some parametric value. ∎

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Distance in Space

- Distance from a point to a line
  The distance D from a point Q(not on ℓ) to a line ℓ in space is given by
$$D = \frac{\|u \times \overrightarrow{PQ}\|}{\|u\|}$$
  where u is the directional vector of ℓ and P is any point on ℓ.
- Distance from a point to a plane
  The perpendicular distance D of a point $P_0 = (x_0, y_0, z_0)$ in space to the plane with equation $ax + by + cz + d = 0$ is given by
$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

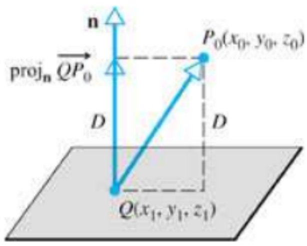

Figure 1.22: The distance from a point to a plane

- Distance between two parallel planes
  Given two parallel planes π_1 and π_2. Then we can have normal vectors with coefficients a, b, c to be the same such that
$$\begin{aligned} \pi_1 : ax + by + cz + d_1 &= 0 \text{ and} \\ \pi_2 : ax + by + cz + d_2 &= 0 \end{aligned}$$
  Then the distance between π_1 and π_2 is given by
$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

■ **Example 1.41** How far is the point $p = (1, 2, 3)$ from the plane with equation $\pi : 3x + 5y - 4z + 37 = 0$.
Solution: Here $(x_0, y_0, z_0) = (1, 2, 3)$, $n = (3, 5, -4)$ and $d = 37$. Thus,
$$\begin{aligned} D &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|3(1) + 5(2) - 4(3) + 37|}{\sqrt{50}} \\ &= \frac{19\sqrt{2}}{5} \text{ unit} \end{aligned}$$
■

■ **Example 1.42** Let $Q = (1, 3, 5)$, $P_0 = (-1, 1, 7)$ and $n = (-1, 1, -1)$. Find the distance between Q and the plane through P_0 and perpendicular to n.
Solution: Let $P = (x, y, z)$ be any arbitrary point on a plane which contain a point $P_0 = (-1, 1, 7)$ and perpendicular to n; that is, $n \cdot \overrightarrow{P_0P} = 0$. So
$$\begin{aligned} & n \cdot \overrightarrow{P_0P} = 0 \\ & \Rightarrow (-1, 1, -1) \cdot ((x, y, z) - (-1, 1, 7)) = 0 \\ & \Rightarrow -x + y - z = 1 + 1 - 7 \\ & \Rightarrow -x + y - z = -5 \end{aligned}$$

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Therefore, $-x+y-z=-5$ is the equation of the plane and $\|n\|=\sqrt{1+1+1}=\sqrt{3}$, $Q-P_0=(2,2,-2)$ and $(Q-P_0)\cdot n=-2+2+2=2$. Hence the distance D is

$$\begin{aligned} D &= \frac{|-1(1)+1(3)+(-1)5+5|}{\sqrt{3}} \\ &= \frac{2}{\sqrt{3}} \end{aligned}$$

Therefore, the distance between Q and the plane is $\frac{2}{\sqrt{3}}$. ■

1.6 Vector Space and Subspace

Definition 1.6.1 Let F be a set of numbers. Then F is said to be a field under the usual addition '+' and scalar multiplication '$\cdot$' denoted by $(F,+,\cdot)$ if it satisfies the following conditions:

1. If x and y are elements of F, then $x+y$ and xy are also elements of F.
2. If x is an element of F, then $-x$ is also an element of F. Furthermore, if $x\neq 0$, then x^{-1} is also an element of F.
3. 0 and 1 are elements of F.

■ **Example 1.43** The set of all real numbers R, rational numbers Q and the set of all complex numbers C are fields but not integers Z because if $x\in Z$, but $x^{-1}\notin Z$. ■

The essential thing about a field is that its elements can be added and multiplied and the results are also elements of the field. Moreover, every element can be divided by a non-zero element.

Definition 1.6.2 A vector space V over a field F is a set of non empty objects which can be added and can be multiplied by elements of F, it satisfies the following properties.

1. Closure
   For any $v,w\in V$, we have $v+w\in V$.
2. Associativity
   For any $u,v,w\in V$, then $(u+v)+w=u+(v+w)$.
3. Commutativity
   For $v,w\in V$, we have $v+w=w+v$
4. Existence of Identity
   There is an element of V, denoted by O(called the zero element), such that $0+u=u=u+0$ for all elements u of V.
5. Existence of Inverse
   For $u\in V$, there exists $-u\in V$ such that $u+(-u)=0=(-u)+u$.
6. For $u\in V$ and $\alpha\in F$, then $\alpha u\in V$.
7. For $v,w\in V$ and $\alpha\in F$, then $\alpha(v+w)=\alpha v+\alpha w$.
8. For $u\in V$ and $\alpha,\beta\in F$, $(\alpha+\beta)u=\alpha u+\beta u$.
9. For $u\in V$ and $\alpha,\beta\in F$, then $(\alpha\beta)u=\alpha(\beta u)$.
10. For all $u\in V$, then $1u=u$ since $1\in F$.

It is important to realize that a vector space consists of four entities: a set of vectors, a set of scalars, and two binary operations. When you refer to a vector space be sure all four entities are clearly stated or understood. Unless stated otherwise, assume that the set of scalars is the set of real numbers.

■ **Example 1.44** Show that $R^3=\{(x,y,z)|x,y,z\in R\}$ with addition defined as addition of points and scalar multiplication of points by scalars is a vector space over R.
Solution: Let $(x_1,y_1,z_1),(x_2,y_2,z_2),(x_3,y_3,z_3)\in R^3$, and α and $\beta\in R$. Then

1. $(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)\in R^3$
   Closure is hold.

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2.

$$\begin{aligned}((x_1,y_1,z_1)+ \quad & (x_2,y_2,z_2))+(x_3,y_3,z_3)\\ &=(x_1+x_2,y_1+y_2,z_1+z_2)+(x_3,y_3,z_3)\\ &=((x_1+x_2)+x_3,(y_1+y_2)+y_3,(z_1+z_2)+z_3)\\ &=(x_1+(x_2+x_3),y_1+(y_2+y_3),z_1+(z_2+z_3))\\ &=(x_1,y_1,z_1)+(x_2+x_3,y_2+y_3,z_2+z_3)\\ &=(x_1,y_1,z_1)+((x_2,y_2,z_2)+(x_3,y_3,z_3))\end{aligned}$$

Addition is associative

3.

$$\begin{aligned}(x_1,y_1,z_1)+(x_2,y_2,z_2) &= (x_1+x_2,y_1+y_2,z_1+z_2)\\ &= (x_2+x_1,y_2+y_1,z_2+z_1)\\ &= (x_2,y_2,z_2)+(x_1,y_1,z_1)\end{aligned}$$

Addition is commutative

4. $(x_1,y_1,z_1)+(0,0,0)=(x_1,y_1,z_1)=(0,0,0)+(x_1,y_1,z_1)$
Their existence identity

5.

$$\begin{aligned}(x_1,y_1,z_1)+(-x_1,-y_1,-z_1) &= (x_1,y_1,z_1)-(x_1,y_1,z_1)\\ &= 0\end{aligned}$$

6. $\alpha(x_1,y_1,z_1)=(\alpha x_1,\alpha y_1,\alpha z_1)\in R^3$
7. $\alpha((x_1,y_1,z_1)+(x_2,y_2,z_2))=\alpha(x_1,y_1,z_1)+\alpha(x_2,y_2,z_2)$
8.

$$\begin{aligned}(\alpha+\beta)(x_1,y_1,z_1) &= ((\alpha+\beta)x_1,(\alpha+\beta)y_1,(\alpha+\beta)z_1)\\ &= (\alpha x_1+\beta x_1,\alpha y_1+\beta y_1),\alpha z_1+\beta z_1))\\ &= \alpha(x_1,y_1,z_1)+\beta(x_1,y_1,z_1)\\ &= \alpha(x_1,y_1,z_1)+\beta(x_1,y_1,z_1)\end{aligned}$$

9.

$$\begin{aligned}(\alpha\beta)(x_1,y_1,z_1) &= ((\alpha\beta)x_1,(\alpha\beta)y_1,(\alpha\beta)z_1)\\ &= (\alpha(\beta x_1),\alpha(\beta y_1),\alpha(\beta z_1))\\ &= \alpha(\beta x_1,\beta y_1,\beta z_1)\\ &= \alpha(\beta(x_1,y_1,z_1))\end{aligned}$$

10. $1(x_1,y_1,z_1)=(1x_1,1y_1,1z_1)=(x_1,y_1,z_1)$

Hence, R^3 is a vector space over R. ■

■ **Example 1.45** R is a vector space over it self. ■

■ **Example 1.46** R^n with standard operations is a vector space. ■

■ **Example 1.47** All polynomials of degree 2 or less is a vector space. ■

■ **Example 1.48** All 3×4 matrices are vector space. ■

■ **Example 1.49** The set defined by $S=\{(x,y,z)|x.y.z\in Q\}$ is not a vector space over R because if we take $\alpha=\sqrt{2}\in R$ and $v=(1,3,0)\in S$, then we can see that αv is not in S. ■

Definition 1.6.3 Suppose V is a vector space over F and W is a non-empty subset of V. If, under the addition and scalar multiplication that is defined on V, W is also a vector space then we call W a subspace of V.

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Theorem 1.6.1 A subset W of a vector space V is called a subspace of V if
1. W is closed under addition; that is, if $v, w \in W$, then $v + w \in W$
2. W is closed under scalar multiplication; that is, if $v \in W$ and $a \in F$, then $av \in W$.
3. W contains the additive identity 0.

Then as $W \subseteq V$, properties of vector space are satisfied for the elements of W. Hence W itself is a vector space over F. We call W a subspace of V.

■ **Example 1.50** V and $\{0\}$ are the trivial subspaces of any vector space V. ■

■ **Example 1.51** Let $V = R^3$ and $W = \{(x,0,0)|x \in R\}$, show that W is a subspace of V.
Solution: Let $v, w \in W$. Then $v = (x_1,0,0)$, $w = (x_2,0,0)$ for some $x_1, x_2 \in R$. So
1. $v + w = (x_1,0,0) + (x_2,0,0) = (x_1 + x_2,0,0) \in W$
2. Let α be a real number, then $\alpha v = (\alpha x_1,0,0) \in W$ since $\alpha x_1 \in R$
3. $0 = (0,0,0) \in W$ when $x_1 = 0$

Hence, W is a subspace of V. ■

■ **Example 1.52** Let $V = R^4$, A be a fixed vector in R^4 and $W = \{B = (x,y,z,w)|A \cdot B = 0\}$. Show that W is a subspace of R^4.
Solution: Here $W \subseteq R^4$
1. Let $B, C \in W$, then $B \cdot A = 0$ and $C \cdot A = 0$. So, $(B + C) \cdot A = B \cdot A + B \cdot A = 0$. Hence $B + C \in W$.
2. Let $\alpha \in R$, then $\alpha B \cdot A = \alpha(B \cdot A) = \alpha 0 = 0$. Hence $\alpha B \in W$.
3. $(0,0,0,0) \cdot A = 0$, then $(0,0,0,0) \in W$.

Therefore, W is a subspace of R^4. ■

■ **Example 1.53** The set of all lines passing through the origin, $L = \{ax + by = 0, a, b \in R\}$ is a subspace of vector space $V = R^2$.
Solution: From the the theorem
1. Let $L_1 = \{ax_1 + by_1 = 0, a, b \in R\}$ and $L_2 = \{ax_2 + by_2 = 0, a, b \in R\}$, then

$$L_1 + L_2 = \{a(x_1 + x_2) + b(y_1 + y_2) = 0, a, b \in R\} \in L$$

2. Let $L_1 = \{ax_1 + by_1 = 0, a, b \in R\}$ and $cL_1 = \{c(ax_1 + by_1) = 0, a, b \in R\} \in L$
3. $L = \{a(0) + b(0) = 0, a, b \in R\}$, this implies that the line is passes through the origin.

Therefore, L is the subspace of V. ■

Theorem 1.6.2 The intersection of the subspaces W_1 and W_2 of a vector space V is also a subspace.
Proof: Here W_1 and W_2 are subspace of V. Now we have to show that the intersection is also a subspace.
1. Let $x, y \in W_1 \cap W_2$, then $x, y \in W_1$ and $x, y \in W_2$. But since we know that W_1 is a subspace, $x + y \in W_1$. Similarly, one can show that $x + y \in W_2$ and therefore $x + y \in W_2$. So, $W_1 \cap W_2$ is indeed closed under addition.
2. Let $\alpha \in F$, then $\alpha x \in W_1$ since W_1 is a subspace. Similarly, $\alpha x \in W_2$. From this $\alpha x \in W_1 \cap W_2$ follows. So, $W_1 \cap W_2$ is closed under scalar multiplication.
3. $0 \in W_1$ since W_1 is a subspace. Similarly, $0 \in W_2$. Therefore, $0 \in W_1 \cap W_2$.

Therefore, $W_1 \cap W_2$ is also a subspace.

1.7 Linear Dependence and Independence

Definition 1.7.1 A vector w in R^n is said to be a linear combination of the vectors $v_1, v_2, \cdots, v_k$ in R^n if w can be expressed in the form

$$w = c_1v_1 + c_2v_2 + \cdots + c_kv_k \tag{1.7}$$

The scalars $c_1, c_2, \cdots, c_k$ are called the coefficients in the linear combination. In the case where $k = 1$, equation(1.7) becomes $w = c_1v_1$, so to say that w is a linear combination of v_1 is the same as saying that w is a scalar multiple of v_1.

■ **Example 1.54** Write $(5,6)$ as a linear combination of the vectors $(2,1)$ and $(0,4)$ if it possible.

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Solution: Suppose there exist numbers α and β. Such that

$$\alpha(2,1)+\beta(0,4)=(5,6)$$

Then $2\alpha=5$ and $\alpha+4\beta=6$

$$\Rightarrow \alpha=\frac{5}{6} \text{ and } \beta=\frac{6-\alpha}{4}=\frac{6-\frac{5}{2}}{4}=\frac{7}{8}$$

Therefore, we can write $(5,6)$ as a linear combination of the vectors $(2,1)$ and $(0,4)$ as

$$\frac{5}{2}(2,1)+\frac{7}{8}(0,4)=(5,6)$$

■

Definition 1.7.2 Let $v_1,v_2,\cdots,v_n$ be elements of a vector space V over F. Let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be elements of F. Then an expression of the form

$$\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_nv_n$$

is called a linear combination of $v_1,v_2,\cdots,v_n$.

■ **Example 1.55** Write $(5,6)$ as a linear combination of the vectors $(2,1)$ and $(0,4)$ if it possible.
Solution: Suppose there exist numbers α and β. Such that

$$\alpha(2,1)+\beta(0,4)=(5,6)$$

Then $2\alpha=5$ and $\alpha+4\beta=6$

$$\Rightarrow \alpha=\frac{5}{6} \text{ and } \beta=\frac{6-\alpha}{4}=\frac{6-\frac{5}{2}}{4}=\frac{7}{8}$$

Therefore, we can write $(5,6)$ as a linear combination of the vectors $(2,1)$ and $(0,4)$ as

$$\frac{5}{2}(2,1)+\frac{7}{8}(0,4)=(5,6)$$

■

Definition 1.7.3 Let $v_1,v_2,\cdots,v_n$ be elements of a vector space V over F. Then vectors are called

1. Linearly Dependent if there exist scalers $\alpha_1,\alpha_2,\cdots,\alpha_n$ not all zero such that $\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_nv_n=0$.
2. Linearly Independent if $\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_nv_n=0$ implies $\alpha_1=\alpha_2=\cdots=\alpha_n=0$

■ **Example 1.56** Show that $(0,1,2),(4,-1,2),(-8,2,-4)$ are linearly dependent.
Solution: Suppose $\alpha,\beta,\gamma\in R$, such that

$$\alpha(0,1,2)+\beta(4,-1,2)+\gamma(-8,2,-4)=0$$

Then

$$\begin{aligned} 4\beta-8\gamma &= 0 \\ \alpha-\beta+2\gamma &= 0 \\ 2\alpha+2\beta-4\gamma &= 0 \end{aligned}$$

From the first equation, $\beta=2\gamma\Rightarrow\beta=4$. Substituting this in the second gives $\alpha=0$. Hence for all number γ,

$$0(0,1,2)+2\gamma(4,-1,2)+\gamma(-8,2,-4)=0$$

In particular if $\gamma=2$, we have

$$0(0,1,2)+4(4,-1,2)+2(-8,2,-4)=0$$

Therefore the vectors are linearly dependent because all the values of α, β and γ are not zero. ■

Any two vectors v and w in R^3 are linearly dependent if and only if they lie on the same line through the origin. Similarly, any three vectors u, v and w in R^3 are linearly dependent if and only if they lie on the same plane through the origin.

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Remark 1.7.1
- Suppose 0 is one of the vectors $v_1, v_2, \cdots, v_n$, say $v_1 = 0$. Then the vectors must be linearly dependent, because we have the zero value linear combination where the coefficient of of v_1 is non zero.
- Suppose v is a non zero vector. Then v,by itself, is linearly independent, because $kv = 0$; $v \neq 0$ implies $k = 0$.
- Suppose two of the vectors $v_1, v_2, \cdots, v_n$ are equal or one is a scalar multiple of the other. Then the vectors must be linearly dependent.
- Two vectors v_1 and v_2 are linearly dependent if and only if one of them is a multiple of the other.
- The set $v_1, v_2, \cdots, v_n$ is linearly independent, then any rearrangement of vectors $v_1, v_2, \cdots, v_n$ is also linearly independent.
- If a set S of vectors is linearly independent, then any subset of S is linearly independent. Alternatively, if S contains a linearly dependent subset, then S is linearly dependent.

1.8 Bases and dimension of a vector space

Definition 1.8.1 Let V be any vector space over a field F and let the set $S = \{v_1, v_2, \cdots, v_n\}$ be a set of vectors in V. Then

1. S is said to be span or generate of V if each element of V is a linear combinations of elements is S.
2. S is called a basis for V if S is a linearly independent set and it spans V.
3. The dimension of V is said to be n ($dimV = n$) if V has basis consisting of n elements.

■ **Example 1.57** Show that $\{(1,2), (-2,1)\}$ form a basis for R^2.
Solution: First, we have to show $(1,2)$ and $(-2,1)$ are linearly independent. Suppose $\alpha(1,2) + \beta(-2,1) = (0,0)$ for some real numbers α and β. Then

$$\begin{aligned} \alpha - 2\beta &= 0 \\ 2\alpha + \beta &= 0 \end{aligned}$$

Multiplying the first equation by one, the second by two and adding two equations gives $4\alpha = 0$; that is, $\alpha = 0$. Substituting $\alpha = 0$ in one of the equations gives $\beta = 0$. Therefore $(1,2)$ and $(-2,1)$ are linearly independent.
Next we have to show $\{(1,2), (-2,1)\}$ spans any element of R^2. Let (x,y) be an arbitrary element of R^2. Suppose

$$\alpha(1,2) + \beta(-2,1) = (x,y)$$

Then

$$\begin{aligned} \alpha - 2\beta &= x \\ 2\alpha + \beta &= y \end{aligned}$$

Then $5\alpha = x + 2y$; that is, $\alpha = \frac{x+2y}{5}$. Substituting the value of α in one of the equations gives $\beta = \frac{y-2x}{5}$. Hence

$$(x,y) = \frac{x+2y}{5}(1,2) + \frac{y-2x}{5}(-2,1)$$

That is any element in R^2 can be written as a linear combination of $(1,2)$ and $(-2,1)$. Therefore, $\{(1,2), (-2,1)\}$ form a basis for R^2. ■

Theorem 1.8.1 There exist a basis for every finite dimensional vector space.

Theorem 1.8.2 If V is a vector space and $S = \{v_1, v_2, \cdots, v_n\}$ is a basis of V, then all the other basis of V has n elements.

■ **Example 1.58** Find a basis for R^3 containing the vectors $\{(1,1,1), (0,1,1)\}$.
Solution: Since neither is a linear combination of the other the vectors are linearly independent. But

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since dim $R^3 = 3$ this set can not be a basis for R^3. To determine basis, find a vector which is not a linear combination of the vector. But

$$\begin{aligned}[(1,1,1),(0,1,1)] &= \{\alpha(1,1,1)+\beta(0,1,1)|\alpha,\beta \in R\} \\ &= \{(\alpha,\alpha+\beta,\alpha+2\beta)|\alpha,\beta \in R\}\end{aligned}$$

Choose $\alpha = 1$ and $\beta = -1$. Then $(1,1,3) \notin [(1,1,1),(0,1,1)]$. Because the second coordinate is not zero. Therefore $\{(1,1,1),(0,1,1),(1,1,3)\}$ is a basis for R^3. ■

Definition 1.8.2 Let $\{u_1,u_2,\cdots,u_n\}$ be a basis for R^n and let $v \in R^n$ such that

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

Then $(\alpha_1,\alpha_2,\cdots,\alpha_n)$ are called the coordinates of v with respect to the basis $\{u_1,u_2,\cdots,u_n\}$. α_i is called the i^{th} component(coordinate) of v.

■ **Example 1.59** The coordinate of $(2,4,5)$ with respect to the standard unit basis of R^3 are $(2,4,5)$ since

$$(2,4,5) = 2(1,0,0)+4(0,1,0)+5(0,0,1) = 2i+4j+5k$$

■

■ **Example 1.60** Find the coordinates of $(3,4)$ with respect to the basis $\{(1,2),(-2,1)\}$.
Solution: Let $(3,4) = \alpha(1,2)+\beta(-2,1)$. Then

$$\begin{aligned}\alpha &= \frac{11}{5} \\ \beta &= \frac{-2}{5}\end{aligned}$$

Therefore $(\frac{11}{5},\frac{-2}{5})$ are the coordinates of $(3,4)$ with respect to the given basis. ■

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1.9 Exercise

1. If u and v are perpendicular unit vectors, then find $\|3u-2v\|$.

   Ans. $\sqrt{13}$

2. Let u, v and w are vectors in R^3 such that $u+v=w$ and $\|u\|=2$, $\|v\|=3$ and $\|w\|=4$. Then find the angle between vector u and v.

   Ans. $\theta=\cos^{-1}\frac{1}{4}$

3. If $\|v\|=6$, $\|w\|=8$ and $v+tw$ and $v-tw$ are orthogonal, then what is the value of t.

   Ans. $t=\pm\frac{3}{4}$

4. Given two vectors v and w such that $\|v\|=3$, $\|w\|=4$ and $\|v-w\|=6$, then find
   (a) $\|v+w\|$
   (b) The cosine of the angle between the two vectors v and w.

   Ans. $\sqrt{14}$ and $\theta=\cos^{-1}-\frac{11}{24}$

5. Suppose v is perpendicular to w and $\|v\|=\|w\|=1$. If $u=\alpha v+\beta w$, show that $\|u\|=\sqrt{\alpha^2+\beta^2}$.
6. If the angle between any two of u, v and w in R^3 space is 60 and $\|u\|=4$, $\|v\|=2$ and $\|w\|=6$, then find $\|u+v\|$ and $\|u+v+w\|$.

   Ans. $\sqrt{28}$ and 10

7. If $u=(1,1,2)$, $v=(4,k,-3)$ and $w=(3,2,-1)$ are coplanar and k is any constant, then find $\text{Proj}_u v$ and $\text{Comp}_w u+2v$.
8. Find the angles that the vector $2i+3j-2k$ makes with the coordinate axes.

   Ans. $\alpha=\cos^{-1}(\frac{2}{\sqrt{17}})$, $\beta=\cos^{-1}(\frac{3}{\sqrt{17}})$ and $\gamma=\cos^{-1}(\frac{-2}{\sqrt{17}})$

9. Let l be the line passing through the points $p_0=(0,2,-1)$ and $p_1=(-1,1,3)$ in space. Then find the parametric and symmetric form of the equation of a line.

   Ans. $\left.\begin{array}{r} x=-t \\ y=-t+2 \\ z=4t-1 \end{array}\right\}$ parametric form and $\frac{x}{-1}=\frac{y-2}{-1}=\frac{z+1}{4}$ symmetric form of a line

10. Find the equation of a plane containing the points $P=(5,-2,3)$, $Q=(1,-1,0)$ and $R=(2,-1,3)$.

    Ans. $3x+9y-z=6$

11. Find the intersection of the line $2x=t$, $y+1=3t$ and $3z+1=4t$, where $t\in R$ and the plane $\pi: 2x+y-3z=6$.

    Ans. No intersection

12. Determine the area of the triangle formed by the vectors $v=2i+4j+6k$ and $w=3i+5j$.

    Ans. 7

13. Determine the volume of the parallelepiped formed by the vectors $u=2i+3j+5k$, $v=3i+j+8k$ and $w=2i+6k+5k$.

    Ans. 3

14. Find the area of the parallelogram with diagonals $v=2i+3j-k$ and $w=3i+2j-4k$.

    Ans. 10

15. Suppose that a line with parametric equation is $\frac{x-1}{1}=\frac{y-1}{3}=\frac{z+5}{3}$ is perpendicular to a plane passing through the point $(3,2,4)$ in space,then
    (a) Find the equation of the plane.

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(b) Find the distance of the resulting plane from the origin.

Ans. $x+3y+3z-21=0$ and $\frac{11}{\sqrt{19}}$

16. If $v=(1,k,3)$ is a linear combination of $v_1=(1,-3,2)$ and $v_2=(2,-1,1)$, then find the value of k.

Ans. $-\frac{14}{3}$

17. Determine whether the sets $\{(2,6,0), (0,1,1), (1,3,1)\}$ is linearly dependent or independent in the vector space R^3.

Ans. Linearly Independent

18. Show that the set of all lines $L=\{ax+by=0|a,b\in R\}$ passing through the origin is a subspace of the vector space $V=R^2$.
19. Determine whether the sets $\{(4,3,2),(2,1,0),(6,4,2)\}$ are linearly dependent or linearly independent.

Ans. Linearly Dependent

20. Determine whether the sets $\{(1,0,0),(0,1,1),(1,0,1)\}$ form a basis in R^3.

Ans. Basis

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